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Basic Analog of Fourier Series on a q-Linear Grid

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$f(x)$ by $\delta f(x)=f(q^{1/2}x)-f(q^{-1/2}x)$. The q-linear initial value problem $\frac{\delta f}{\delta x}=$ $\lambda f(x)$, $f(0)=1$, has two entire functions $C_q(z)$ and $S_q(z)$ as linearly independent solutions. The functions $C_q(z)$ and $S_q(z)$ are orthogonal on a discrete set. We consider Fourier expansions in these functions and derive analytic bounds on the roots of $S_q(z)$. © 2001 Academic Press

1. INTRODUCTION

Let $f(x)$ be a function defined for $-\infty<x<\infty$ and let $0<q<1$. The symmetric q-difference operator is defined by

$$\delta f(x)=f(q^{1/2}x)-f(q^{-1/2}x). \tag{1.1}$$

The operator defined in (1.1) can be viewed as the "q-linear" analog of the operator studied in [2]. From (1.1) it is obvious that

$$\frac{\delta f(x)}{\delta x}=\frac{f(q^{1/2}x)-f(q^{-1/2}x)}{x(q^{1/2}-q^{-1/2})}. \tag{1.2}$$

There is a critical relation between this difference operator and the q-integral. The q-integral is defined by

$$\int_0^a f(x)\,d_qx=\sum_{k=0}^{\infty} f(aq^k)\,aq^k(1-q), \tag{1.3}$$

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and

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x. \tag{1.4}$$

From (1.2) and (1.4) it follows that

$$\int_{-1}^{1} \frac{\delta g(x)}{\delta x}\, d_q x = q^{1/2}\{[g(q^{-1/2}) - g(-q^{-1/2})] - [g(0^+) - g(0^-)]\}. \tag{1.5}$$

The classical exponential function $e^{\lambda x}$ is a solution of the initial value problem

$$f'(x) = \lambda f(x), \qquad f(0) = 1; \tag{1.6}$$

the trigonometric functions are then obtained from the real and imaginary parts of $e^{i\lambda x}$. In this paper we will consider a difference analog of (1.6) using the symmetric q-difference operator defined above. We will then define an analog of the classical exponential function and study Fourier expansions in series of "q-linear trigonometric functions." We will find that these difference analogs of sine and cosine are orthogonal in a discrete set.

Throughout this paper we will follow the notation used in [3] which has now become standard.

2. THE q-LINEAR SINE AND COSINE

It is possible to derive the functions to be discussed by taking appropriate limits in a general approach given in [9]. However, we judge it best here to follow M. Rahman [8] and derive them directly. We begin by considering the initial value problem

$$\frac{\delta f(x)}{\delta x} = \lambda f(x), \qquad f(0) = 1. \tag{2.1}$$

If we presume an analytic solution of (2.1), say

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then we find from (2.1) that

$$a_n = \frac{\lambda^n (1-q)^n q^{n^2 - n/4}}{(q; q)_n}, \qquad n = 0, 1, 2, \ldots .$$

On the basis of these calculations and because the initial value problem (2.1) is a q-difference analog of the differential initial value problem satisfied by $\exp \lambda z$, we define

$$\exp_q [\lambda(1-q) z] = \sum_{n=0}^{\infty} \frac{[\lambda(1-q) z]^n q^{(n^2-n)/4}}{(q; q)_n}. \tag{2.2}$$

We then define the q-linear sine and cosine, $S_q(z)$ and $C_q(z)$, by

$$\exp_q iz \doteq C_q(z) + iS_q(z).$$

From (2.2) we get

$$C_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n[n-(1/2)]} z^{2n}}{(q; q^2; q^2)_n}, \tag{2.3}$$

$$S_q(z) = \frac{z}{1-q} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n[n+(1/2)]} z^{2n}}{(q^2, q^3; q^2)_n}. \tag{2.4}$$

The notation $(a, b; q)_n$ used in (2.3) and (2.4) means $(a; q)_n (b; q)_n$.

As a consequence of (2.1) we have

$$\frac{\delta C_q(\omega z)}{\delta z} = -\frac{\omega}{1-q} S_q(\omega z) \tag{2.5}$$

$$\frac{\delta S_q(\omega z)}{\delta z} = \frac{\omega}{1-q} C_q(\omega z). \tag{2.6}$$

From (2.5) and (2.6) it follows that both $S_q(\omega z)$ and $C_q(\omega z)$ satisfy the second order equation

$$\frac{\delta}{\delta z}\left(\frac{\delta u}{\delta z}\right) + \frac{\omega^2}{(1-q)^2} u = 0. \tag{2.7}$$

3. RELATION WITH q-BESSEL FUNCTIONS AND q-HYPERGEOMETRIC SERIES

$C_q(z)$ and $S_q(z)$ can be written in q-hypergeometric notation as

$$C_q(z) = {}_1\phi_1\left(\begin{matrix} 0 \\ q \end{matrix}; q^2, q^{1/2} z^2\right), \tag{3.1}$$

$$S_q(z) = \frac{z}{1-q}\, {}_1\phi_1\left(\begin{matrix} 0 \\ q^3 \end{matrix}; q^2, q^{3/2} z^2\right). \tag{3.2}$$

The functions $C_q(z)$ and $S_q(z)$ are related to one of the three known q-analogues of the classical Bessel functions. This is known as the Third Jackson q-Bessel function and as the Hahn–Exton q-Bessel function [4, 5, 10] and is defined as

$$J_\nu^{(1.3)}(z;q) \doteq \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} z^\nu {}_1\phi_1\left(\begin{matrix}0\\ q^{\nu+1}\end{matrix};q,qz^2\right). \tag{3.3}$$

Thus, we have

$$C_q(z) = q^{-3/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{-1/2}^{(1.3)}(q^{-3/4}z;q^2), \tag{3.4}$$

$$S_q(z) = q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J_{1/2}^{(1.3)}(q^{-1/4}z;q^2). \tag{3.5}$$

There is a simple transformation that will be critical further on. It is

$$(c;q)_\infty\, {}_1\phi_1\left(\begin{matrix}0\\ c\end{matrix};q,z\right) = (z;q)_\infty\, {}_1\phi_1\left(\begin{matrix}0\\ z\end{matrix};q,c\right). \tag{3.6}$$

Thus from (3.1) and (3.2) applying (3.6) we get

$$S_q(z) = \frac{z}{1-q}\frac{(q^{3/2}z^2;q^2)_\infty}{(q^3;q^2)_\infty}\, {}_1\phi_1\left(\begin{matrix}0\\ q^{3/2}z^2\end{matrix};q^2,q^3\right), \tag{3.7}$$

$$C_q(z) = \frac{(q^{1/2}z^2;q^2)_\infty}{(q;q^2)_\infty}\, {}_1\phi_1\left(\begin{matrix}0\\ q^{1/2}z^2\end{matrix};q^2,q\right). \tag{3.8}$$

It is known that the q-Bessel functions $J_\nu^{(1.3)}(x;q)$ have only real roots and that the roots are simple. Thus the roots of $C_q(z)$ and $S_q(z)$ are real and simple. Also, since $C_q(z)$ and $S_q(z)$ are respectively even and odd, it follows that the roots of $C_q(z)$ and $S_q(z)$ are symmetric. We will denote the positive roots of $S_q(z)$ by $w_1 < w_2 < w_3 < \cdots$. Further, the functions $C_q(z)$ and $S_q(z)$ are entire of order zero.

4. ORTHOGONALITY

Suppose that $u_1(z)$ and $u_2(z)$ are solutions of the second order difference equation

$$\frac{\delta}{\delta z}\left(\frac{\delta u}{\delta z}\right) + \lambda_1, u_i = 0, \qquad i=1,2,\ \lambda_1 \neq \lambda_2. \tag{4.1}$$

Then we have

$$(\lambda_1-\lambda_2)\,u_1(z)\,u_2(z)=u_1(z)\,\frac{\delta}{\delta z}\left(\frac{\delta u_2(z)}{\delta z}\right)-u_2(z)\,\frac{\delta}{\delta z}\left(\frac{\delta u_1(z)}{\delta z}\right). \tag{4.2}$$

Define the q-Wronskian by

$$W(u_1(z),u_2(z))=u_1(q^{-1/2}z)\,\frac{\delta u_2(z)}{\delta z}-u_2(q^{-1/2}z)\,\frac{\delta}{\delta z}\,u_1(z). \tag{4.3}$$

Using (4.3), (4.2) may be written as

$$(\lambda_1-\lambda_2)\,u_1(z)\,u_2(z)=\frac{\delta}{\delta z}\,(W(u_1(z),u_2(z)). \tag{4.4}$$

Computing the q-integral of (4.4) and using (1.5) gives

$$\begin{aligned}(\lambda_1-\lambda_2)&\int_{-1}^{1}u_1(z)\,u_2(z)\,d_qz\\&=q^{1/2}\{W(u_1(q^{-1/2}),u_2(q^{-1/2}))-W(u_1(-q^{-1/2}),u_2(-q^{-1/2}))\}\\&\quad-q^{1/2}\{W(u_1(0^+),u_2(0^+))-W(u,(0^-),u_2(0^-))\}.\end{aligned} \tag{4.5}$$

In particular, suppose we take in (4.5)

$$u_1(z)=C_q(q^{1/2}w_1z),\qquad u_2(z)=S_q(qw_2z).$$

Then the right side of (4.5) vanishes because the integrand in (4.5) is an odd function. If, again, we make the choice $u_1(z)=C_q(q^{1/2}w,z)$, $u_2(z)=C_q(q^{1/2}w_2z)$, then (4.5) becomes

$$\begin{aligned}(w_1-w_2)&\int_{-1}^{1}C_q(q^{1/2}w_1z)\,C_q(q^{1/2}w_2z)\,d_qz\\&=\frac{2q}{1-q}\,[w_1C_q(q^{-1/2}w_2)\,S_q(w_1)-w_2C_q(q^{-1/2}w_1)\,S_q(w_2)].\end{aligned} \tag{4.6}$$

Last, if we choose $u_1(z)=S_q(qw_1z)$, $u_2(z)=S_q(qw_2z)$ then (4.5) becomes

$$\begin{aligned}(\omega_1-\omega_2)&\int_{-1}^{1}S_q(qw_1z)\,S_q(qw_2z)\,d_qz\\&=\frac{2q^{3/2}}{1-q}\,[w_2C_q(q^{1/2}w_2)\,S_q(w_1)-w_1C_q(q^{1/2}w_1)\,S_q(w_2)].\end{aligned} \tag{4.7}$$

From the right sides of (4.6) and (4.7) it is clear that both integrals vanish if ω_1 and ω_2 are taken to be roots of $S_q(z)$. A calculation with L'Hopital's rule gives the value of the integrals in (4.6) and (4.7) when $\omega_1 = \omega_2$. We then have the following orthogonality relation.

THEOREM 4.1. *Let w and w' be roots of $S_q(z)$. Then*

$$\int_{-1}^{1} C_q(q^{1/2}wx)\, C_q(q^{1/2}w'x)\, d_qx = \begin{cases} 0, & \text{if } w \neq w' \\ 2, & \text{if } w = w' = 0 \\ \mu(w), & \text{if } w = w' \neq 0 \end{cases}$$

$$\int_{-1}^{1} S_q(qwx)\, S_q(qw'x)\, d_qx = \begin{cases} 0 & \text{if } w \neq w' \text{ or } w = w' = 0 \\ q^{-1/2}\mu(\omega) & \text{if } w = w' \neq 0, \end{cases}$$

where $\mu(w) = (1-q)\, C_q(q^{1/2}w)(\partial/\partial w)\, S_q(w)$.

An analog of the trigonometric identity

$$\sin^2 x + \cos^2 x = 1$$

may be proved for $S_q(z)$ and $C_q(z)$. We begin by noting that if $f(x)$ is continuous at $x=0$ and if $\delta f(x) = 0$ for $-\infty < x < \infty$ then $f(x)$ is constant. This follows from the fact that $\delta f(x) = 0$ for $-\infty < x < \infty$ implies that $f(x) = f(qx)$ for $-\infty < x < \infty$. By iteration, $f(x) = f(q^n x)$ for $n = 0, 1, \ldots$. Taking a limit on n gives $f(x) = f(0)$ for $-\infty < x < \infty$.

Calculating the Wronskian defined by (4.3) for the functions $u_1(z) = S_q(z)$, $u_2 = C_q(z)$ we find

$$W(S_q(z), C_q(z)) = \frac{1}{q-1}\,(S_q(q^{-1/2}z)\, S_q(z) + C_q(q^{-1/2}z)\, C_q(z)).$$

Then calculating the difference we get $\delta W(S_q(z), C_q(z)) \equiv 0$. Thus we may conclude that

$$W(S_q(z), C_q(z)) = W(S_q(0), C_q(0)) = \frac{1}{q-1},$$

that is,

$$S_q(q^{-1/2}z)\, S_q(z) + C_q(q^{-1/2}z)\, C_q(z) = 1. \tag{4.8}$$

5. THE ROOTS OF $S_q(z)$

In this section we will derive bounds on the positive roots w_n of $S_q(z)$. In order to obtain bounds valid for all the positive roots w_n, $n=1, 2, \ldots$, we will find it necessary to restrict the range of q. We begin with a preliminary lemma.

LEMMA 5.1. *Define $\alpha_n(q)$ by*

$$\alpha_n(q)=\frac{\log\left[1-\dfrac{q^{2n+1}}{1-q^{2n}}\right]}{2\log q}, \qquad n=1, 2, \ldots .$$

(i) *If $0<q<1$ and $(1-q^2)^2-q^3>0$ then $0<\alpha_n(q)<1$, $n=1, 2, 3, \ldots$.*

(ii) *$(1-q^2)^2-q^3$ has a simple root β_0 in $0<q<1$ and $\beta_0\approx 0.67104$. Thus $(1-q^2)^2-q^3>0$ for $0<q<\beta_0$, and hence $0<\alpha_n(q)<1$ for $n=1, 2, \ldots$, if $0<q<\beta_0$.*

(iii) *If $0<q<\beta_0$ then $\alpha_n(q)=0(q^{2n})$ as $n\to\infty$.*

Proof. (i) $0<\alpha_n(q)$ requires only that

$$0<1-\frac{q^{2n+1}}{1-q^{2n}}.$$

This holds if $1-q^{2n}-q^{2n+1}>0$. However, for $n=1, 2, \ldots$, we have that $1-q^{2n}-q^{2n+1}\geqslant 1-q^2-q^3>0$ if $0<q<3/4$ and certainly then $0<\alpha_n(q)$ if $0<q<\beta_0$. To prove that $\alpha_n(q)<1$ for $n=1, 2, \ldots$, $0<q<\beta_0$, we must prove

$$1-\frac{q^{2n+1}}{1-q^{2n}}>q^2, \qquad n=1, 2, \ldots, 0<q<\beta_0,$$

or, that is, $1-q^2+q^{2n}(q^2-q-1)>0$. But, for $n=1, 2, \ldots$ we have

$$1-q^2+q^{2n}(q^2-q-1)>1-q^2+q^2(q^2-q-1)=(1-q^2)^2-q^3>0$$

for $0<q<\beta_0$.

(ii) This is an elementary calculus exercise.

(iii) This is a consequence of the Taylor expansion of $\alpha_n(q)$. ∎

LEMMA 5.2. *If $(1-q^2)^2>q^3$, then $\operatorname{sgn}[S_q(q^{-m+1/4})]=(-1)^m$, $m=0, 1, 2, \ldots$.*

Proof. Set $z = q^{-m+1/4}$ in (3.7) to get

$$\frac{(q; q^2)_\infty \, S_q(q^{-m+1/4})}{q^{-m+1/4}} = \sum_{k=0}^{\infty} \frac{(-1)^k \, (q^{-2m+2k+2}; q^2)_\infty \, q^{k(k+2)}}{(q^2; q^2)_k}. \tag{5.1}$$

Since $(q^{-2m+2k+2}; q^2)_\infty = 0$ for $k \leqslant m-1$ we have from (5.1)

$$(q; q^2)_\infty \, q^{m-1/4} S_q(q^{-m+1/4}) = \sum_{k=m}^{\infty} \frac{(-1)^k \, (q^{-2m+2k+2}; q^2)_\infty \, q^{k(k+2)}}{(q^2; q^2)_k}.$$

This last series can be written as

$$(-1)^m \sum_{j=0}^{\infty} \frac{(-1)^j \, (q^{2j+2}; q^2)_\infty \, q^{(m+j)(m+j+2)}}{(q^2; q^2)_{m+j}} \doteq (-1)^m \sum_{j=0}^{\infty} (-1)^j \, A_j.$$

We will now prove that $A_{j+1} < A_j$ if $q^3 < (1-q^2)^2$. An easy calculation shows that $A_{j+1} < A_j$ if and only if

$$q^{(2m+2j+3)} < (1-q^{2m+2j+2})(1-q^{2j+2}). \tag{5.2}$$

When $j = 0$, (5.2) becomes

$$q^{2m+3} < (1-q^{2m+2})(1-q^2), \tag{5.3}$$

which is true for $m = 0, 1, \ldots,$ if $q^3 < (1-q^2)^2$. Since (5.3) holds for $q^3 < (1-q^2)^2$, we have

$$q^{2m+2j+3} < q^{2m+3} < (1-q^{2m+2})(1-q^2) < (1-q^{2m+2j+2}) \cdot (1-q^{2j+2}).$$

Since $A_{j+1} < A_j$, $j = 0, 1, \ldots,$ we have

$$\operatorname{sgn}(-1)^m \sum_{j=0}^{\infty} (-1)^j \, A_j = (-1)^m$$

if $q^3 < (1-q^2)^2$, that is, for $0 < q < \beta_0$. This proves Lemma 5.2. ∎

Lemma 5.2 says that $S_q(z)$ has an odd number of roots in each interval (q^{-m+1}, q^{-m}), $m = 0, 1, \ldots$. The next lemma refines this statement.

LEMMA 5.3. *If* $(1-q^2)^2 > q^3$, *then*

$$\operatorname{sgn} S_q(q^{-m+(1/4)+\alpha_m(q)}) = (-1)^{m-1}, \qquad m = 0, 1, \ldots .$$

Proof. For convenience write α_m for $\alpha_m(q)$. In (3.7) set $z=q^{-m+\alpha_m+1/4}$. Then

$$S_q(q^{-m+\alpha_m+1/4})=\sum_{n=0}^{m-2}\frac{(-1)^n q^{n(n+2)}(q^{2+2n-2m+2\alpha_m};q^2)_\infty}{(q^2;q^2)_n}$$
$$+\sum_{n=m-1}^{\infty}\frac{(-1)^n q^{n(n+2)}(q^{2+2n-2m+2\alpha_m};q^2)_\infty}{(q^2;q^2)_n}. \tag{5.4}$$

Denote the finite sum in (5.3) by S_1 and the infinite sum by S_2. In S_1 for $0\leqslant n\leqslant m-2$ and $0<\alpha_m<1$ we have

$$\operatorname{sgn}(q^{2+2n-2m+2\alpha_m};q^2)_\infty=(-1)^{m-1-n}.$$

Thus sgn $S_1=(-1)^{m-1}$. In S_2 the change of variable $k=n-m+1$ yields

$$S_2=(-1)^{m-1}\sum_{k=0}^{\infty}(-1)^k A_k,$$

where A_k is given by

$$A_k=\frac{q^{(k+m-1)(k+m+2)}(q^{2k+2\alpha_m};q^2)_\infty}{(q^2;q^2)_{k+m-1}}.$$

Clearly $A_k>0$. We will prove that $A_{k+1}<A_k$, $k=0,1,\ldots$, and thus $\sum_{k=0}^{\infty}(-1)^k A_k>0$.

A short calculation shows that $A_{k+1}<A_k$ reduces to the inequality

$$q^{2k+2m+1}<(1-q^{2k+2\alpha_m})(1-q^{2k+2m}). \tag{5.5}$$

We establish (5.5) by using the following string of inequalities.

$$q^{2k+2m+1}<q^{2m+1}=(1-q^{2\alpha_m})(1-q^{2m})<(1-q^{2\alpha_m+2k})(1-q^{2k+2m}).$$

Thus (5.5) holds and we have that

$$\operatorname{sgn} S_2=(-1)^{m-1}=\operatorname{sgn} S_1.$$

This proves the lemma. ∎

Lemma 5.2 and Lemma 5.3 imply that $S_q(z)$ has an odd number of roots in each interval $(q^{-m+\alpha_m(q)+1/4},q^{-m+1/4})$. We will now prove Theorem 5.1 which states that there is only one root in each such interval and that there are no other positive roots.

THEOREM 5.1. *Let $0<q<\beta_0$ where $\beta_0 \approx .67104$ is the root of $(1-q^2)^2-q^3=0$ in $0<q<1$. If $\omega_1<\omega_2<\cdots$ are the positive roots of $S_q(z)$ then*

$$w_k = q^{-k+1/4+\epsilon_k}, \qquad 0<\epsilon_k<\alpha_k(q), k=1, 2, ..., \tag{5.6}$$

where $\alpha_k(q)$ is as in Lemma 5.1. *There are no positive roots other than those of the form stated in* (5.6).

Proof. After Lemmas 5.2 and 5.3 it is known that $S_q(z)$ has roots of the form (5.6). We need to prove that there are no other positive roots. To accomplish this we will apply a formula of Jensen [1]. Define

$$F(z) = \frac{(1-q)\, S_q(z)}{z}.$$

From (2.4) we have

$$F(q^{-m-1/4}e^{i\theta}) = \frac{q^{-m^2}(-1)^m\, e^{2im\theta}}{(q^2; q^2)_m\, (q^3; q^2)_m} \sum_{k=-m}^{\infty} \frac{(-1)^k q^{k^2} e^{2ik\theta}}{(q^{2m+2}; q^2)_k\, (q^{2m+3}; q^2)_k}. \tag{5.7}$$

The roots of $F(z)$ by Lemma 5.2 and 5.3 must include $\pm\omega_k = \pm q^{-k+(1/4)+\epsilon_k}$, $k=1, ..., m$. We want to prove that these are the only roots in $|z|<q^{-m-1/4}$. Suppose there are other roots $\pm\lambda_k$, $k=1, ..., P_m$, with $\lambda_k>0$. Jensen's formula then gives ($F(0)=1$)

$$2\sum_{k=1}^{m} \log \frac{q^{-m-1/4}}{w_k} + 2\sum_{k=1}^{P_m} \log \frac{q^{-m-1/4}}{\lambda_k} = \frac{1}{2\pi}\int_0^{2\pi} \log |F(q^{-m-1/4}e^{i\theta})|\, d\theta. \tag{5.8}$$

Write $\omega_k = q^{-k-1/4+\epsilon_k}$ in the left side of (5.8) to get

$$\begin{aligned} &2\sum_{k=1}^{m} \log \frac{q^{-m-1/4}}{w_k} + 2\sum_{k=1}^{P_m} \log \frac{q^{-m-1/4}}{\lambda_k} \\ &\qquad = -m^2 \log q - 2(\log q) \sum_{k=1}^{m} \epsilon_k + 2\sum_{k=1}^{P_m} \log \frac{q^{-m-1/4}}{\lambda_k}. \end{aligned} \tag{5.9}$$

Also, from (5.7)

$$\begin{aligned} \log |F(q^{-m-1/4}e^{i\theta})| = &-m^2 \log q - \log(q^2; q^2)_m - \log(q^3; q^2)_m \\ &+ \log \left| \sum_{k=-m}^{\infty} \frac{(-1)^k q^{k^2} e^{2ik\theta}}{(q^{2m+2}; q^2)_k\, (q^{2m+3}; q^2)_k} \right|. \end{aligned} \tag{5.10}$$

Note that, for the sum in (5.10),

$$\lim_{m\to\infty}\sum_{k=-m}^{\infty}\frac{(-1)^k q^{k^2}e^{2ik\theta}}{(q^{2m+2};q^2)_k\,(q^{2m+3};q^2)_k}$$
$$=\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}e^{2ik\theta}=(q^2,qe^{2i\theta},qe^{-2i\theta};q^2)_\infty. \qquad (5.11)$$

The last equality is an application of the Jacobi triple product identity [3]. The limit in (5.11) can be shown to be uniform in θ. Because of (5.11) we can write

$$\lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\sum_{k=-m}^{\infty}\frac{(-1)^k\,q^{k^2}e^{2ik\theta}}{(q^{2m+2};q^2)_k\,(q^{2m+3};q^2)_k}\right|d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\log|(q^2,qe^{2i\theta},qe^{-2i\theta};q^2)_\infty|\,d\theta.$$

Now,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|(q^2,qe^{2i\theta},qe^{-2i\theta};q^2)_\infty|\,d\theta$$
$$=\log(q^2;q^2)_\infty+\sum_{j=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|1-q^{2j+1}e^{2i\theta}|\,d\theta$$
$$+\sum_{j=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\log|1-q^{2j+1}e^{-2i\theta}|\,d\theta.$$

We applied uniform convergence to exchange integral and sum above. But all these integrals above vanish (the mean value theorem for harmonic functions). Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\log|(q^2,qe^{2i\theta},qe^{-2i\theta};q^2)_\infty|\,d\theta=\log(q^2;q^2)_\infty. \qquad (5.12)$$

Using (5.12) and (5.11) in (5.10) we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log|F(q^{-m-1/4}e^{i\theta})|\,d\theta=-m^2\log q-\log(q^3;q^2)_\infty+o(1) \qquad (5.13)$$

as $m\to\infty$.

TABLE 1

k	$q^{-k+(1/4)+\alpha_k}$	ω_k	$q^{-k+1/4}$
1	1.535259783865635	1.541670759157614	1.681792830507429
2	3.307050783152858	3.360658583239728	3.363585661014858
3	6.700423019986493	6.727168071457452	6.727171322029716
4	13.441145639761449	13.45434264385496	13.454342644059432
5	26.902108559366568	26.90868528811886	26.908685288118865
6	53.814084922180745	53.81737057623774	53.817370576237730
7	107.633098663883582	107.6347411524754	107.634741152475461
8	215.268661102948532	215.269482304951	215.269482304950923
9	430.538554014186999	430.5389646099018	430.538964609901846

Equating (5.13) and (5.9) gives

$$-2(\log q)\sum_{k=1}^{m}\epsilon_k+2\sum_{k=1}^{P_m}\log\frac{q^{-m-1/4}}{\lambda_k}+\log(q^3;q^2)_\infty=o(1)$$

$$\text{as } m\to\infty. \tag{5.14}$$

However, because $\epsilon_k=O(q^{2k})$, we have that $\sum_{k=1}^{\infty}\epsilon_k<\infty$. Also,

$$\sum_{k=1}^{P_m}\log\frac{q^{-m-1/4}}{\lambda_k}>\log\frac{q^{-m-1/4}}{\lambda_1}\to\infty.$$

Thus the only way that (5.14) can hold is if the sum involving λ_k is empty. This proves that there are no other roots besides the w_k. ∎

The result of Lemmas 5.2 and 5.3 and Theorem 5.4 can be stated as

THEOREM 5.2. *If $0<q<\beta_0$ where β_0 is the unique root of $(1-q^2)^2-q^3=0$, $0<q<1$, then the roots of $S_q(z)$ satisfy the inequality*

$$q^{-k+\alpha_k+1/4}<w_k<q^{-k+1/4},\qquad k=1,2,\ldots. \tag{5.15}$$

The accuracy of the bounds in (5.15) is illustrated in Table 1 by the numerical calculations of the first nine roots of $S_q(z)$ for $q=.5$ accurate to 15 places.

Remark 1. The restriction $q^3<(1-q^2)^2$, or that is, $0<q<\beta_0$ in Lemmas 5.2 and 5.3 and Theorem 5.2 are necessary to obtain the representation

$$\omega_k=q^{-k+(1/4)+\epsilon_k},\qquad 0<\epsilon_k<\alpha_k(q)$$

for every integer value $k = 1, 2, \ldots$. It is possible to re-write these proofs and get a similar result valid for large values of k and all $0 < q < 1$. That is, for any q, $0 < q < 1$, K exists such that if $k \geqslant K$ then $\omega_k = q^{-k+(1/4)+\epsilon_k}$, $0 < \epsilon_k < \alpha_k(q)$.

Remark 2. Discarding the empty sum in (5.14) we have

$$\sum_{k=1}^{m} \epsilon_k = \frac{\log(q^3; q^2)_\infty}{2 \log(1/q)} + o(1).$$

Taking the limit as $m \to \infty$ gives

$$\sum_{k=1}^{\infty} \epsilon_k = \frac{\log(q^3; q^2)_\infty}{2 \log(1/q)}.$$

An estimate on the growth of ϵ_m is easily obtained since $0 < \epsilon_m < \alpha_m$, and by Taylor's formula:

$$\alpha_m = \frac{\log\left(1 - \dfrac{q^{2m+1}}{1-q^{2m}}\right)}{2 \log q} = -\frac{q^{2m+1}}{2(1-q^{2m}) \log q} \left\{1 + \frac{\theta}{2} \frac{q^{2m+1}}{1-q^{2m}}\right\}$$

where $0 < \theta < (1 - (q^{2m+1}/1 - q^{2m}))^{-1}$. For $0 < q < \beta_0 = .67104\ldots.$ This gives $0 < \epsilon_m < 3.18 q^{2m+1}$. This last bound on ϵ_m immediately implies that

$$\omega_m = q^{-m+1/4+\epsilon_m} = q^{-m+1/4} + O(q^m).$$

6. THE FOURIER COEFFICIENTS

On the basis of the orthogonality established in Theorem 4.1 we may consider formal Fourier expansions of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k C_q(q^{1/2} w_k x) + b_k S_q(q w_k x),$$

with

$$a_k = \frac{1}{\mu_k} \int_{-1}^{1} f(t)\, C_q(q^{1/2} w_k t)\, d_q t,$$

$$b_k = \frac{1}{\mu_k} \int_{-1}^{1} f(t)\, S_q(q w_k t)\, d_q t,$$

where $\mu_k = (1-q)\, C_q(q^{1/2} w_k)\, S'_q(w_k)$.

It is clear that the behavior of μ_k is critical to an analysis of a_k and b_k and to questions of convergence. In this section we will give asymptotic information about μ_k by estimating $S'_q(w_k)$ and by showing that $C_q(q^{1/2}w_k)$ can be eliminated.

THEOREM 6.1.

$$S'_q(w_k) = \frac{2}{1-q}\, q^{-(k-1/2-\epsilon_k)^2} S_k,$$

where $\liminf_{k\to\infty} |S_k| > 0$ *for* $0 < q \leqslant \beta_0$. *Here* β_0 *is the unique root in* $(0, 1)$ *of* $(1-q^2)^2 - q^3 = 0$, $\beta_0 \cong .67104$.

Proof. Computing the derivative of the infinite series for $S_q(z)$ we find

$$(1-q)\, S'_q(w_k) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n\, n q^{n(n+1/2)} w_k^{2n}}{(q^2, q^3; q^2)_n}.$$

Writing $w_k = q^{-k+1/4+\epsilon_k}$ we find $(1-q)\, S'_q(w_k) = q^{-(k-1/2-\epsilon_k)^2} S_k$ where

$$S_k = \sum_{n=0}^{\infty} \frac{(-1)^n\, n q^{(n-k+1/2+\epsilon_k)^2}}{(q^2, q^3; q^2)_n}.$$

We need to prove that $\liminf_{k\to\infty} |S_u| > 0$. Making a change of variable $m = n-k$ in the series for S_k we get

$$(-1)^k S_k = \sum_{m=-k}^{\infty} \frac{(-1)^m\, m q^{(m+1/2+\epsilon_k)^2}}{(q^2, q^3; q^2)_m}.$$

Set

$$F_m(k) = \frac{(-1)^m\, m q^{(m+1/2+\epsilon_k)^2}}{(q^2, q^3; q^2)_m}$$

and let P be a positive integer. Then for $k > 2P+2$,

$$|S_k| \geqslant \left| \sum_{m=-2P-1}^{2P} F_m(k) \right| - \sum_{m=-k}^{-2P-2} |F_m(k)| - \sum_{m=2P+1}^{\infty} |F_m(k)|. \tag{6.1}$$

We now begin a sequence of tedious calculations. Denote the sums on the right side of (6.1) by G_1, G_2, G_3, respectively. We need to require that P be large enough so that if $k > 2P+z$ then $1/2-\epsilon_k > 0$ (recall that $\epsilon_k \to 0$). For G_2 we have

$$G_2 = \sum_{m=-k}^{-2P-2} \frac{|m|\, q^{(m+\epsilon_k+1/2)^2}}{(q^2, q^3; q^2)_m} < \frac{1}{(q^2; q)_\infty} \sum_{m=2P+2}^{\infty} mq^{(m-\epsilon_k-1/2)^2}$$

$$< \frac{1}{(q^2; q)_\infty} \sum_{m=2P+2}^{\infty} mq^{(m-1)^2} = \frac{1}{(q^2; q)_\infty} \sum_{m=0}^{\infty} (m+2P+2)\, q^{(m+2P+1)^2}$$

$$< \frac{q^{(2P+1)^2}}{(q^2; q)_\infty} \sum_{m=0}^{\infty} (m+2p+2)\, q^{m(4P+2)}$$

$$= \frac{q^{(2P+1)^2}}{(q^2; q)_\infty} \frac{[2P+2-(2P+1)\, q^{4P+2}]}{(1-q^{4P+2})^2}.$$

For G_3 we have

$$G_3 = \sum_{m=2P+1}^{\infty} |F_m(k)| < \frac{1}{(q^2; q)_\infty} \sum_{m=2P+1}^{\infty} mq^{(m+\epsilon_k+1/2)^2}$$

$$< \frac{1}{(q^2; q)_\infty} \sum_{m=2P+1}^{\infty} mq^{m^2} < \frac{q^{(2P+1)^2}}{(q^2; q)_\infty} \sum_{m=0}^{\infty} (m+2P+1)\, q^{m(4P+2)}$$

$$= \frac{q^{(2P+1)^2}}{(q^2; q)_\infty} \frac{2P+1-2Pq^{4P+2}}{(1-q^{4P+2})^2}.$$

Last, to estimate G_1, we have

$$G_1 = \sum_{m=-2P-1}^{2P} F_m(k) = \sum_{m=-2P-1}^{2P} \frac{(-1)^m\, mq^{(m+\epsilon_k+1/2)^2}}{(q^2, q^3; q^2)_{m+k}},$$

and

$$\lim_{k\to\infty} \sum_{m=-2P-1}^{2P} F_m(k) = \sum_{m=-2P-1}^{2P} \frac{(-1)^m\, mq^{(m+1/2)^2}}{(q^2, q^3; q^2)_\infty}.$$

Since

$$\sum_{m=-2P-1}^{2P} (-1)^m\, mq^{(m+1/2)^2} = q^{1/4} \sum_{j=0}^{2P} (-1)^j\, (2j+1)\, q^{j(j+1)}$$

we have

$$G_1 = \frac{q^{1/4}}{(q^2; q)_\infty} \left| \sum_{j=0}^{2P} (-1)^j\, (2j+1)\, q^{j(j+1)} \right| + o(1) \qquad \text{as} \quad k \to \infty.$$

Combining the estimates for G_1, G_2, G_3, we have

$$S_k \geqslant \frac{1}{(q^2;q)_\infty}\left\{q^{1/4}\left|\sum_{j=0}^{2P}(-1)^j(2j+1)\,q^{j(j+1)}\right|\right.$$
$$\left.-\frac{q^{(2P+1)^2}}{(1-q^{4P+2})^2}\left[4P+3-(4P+1)\,q^{4P+2}\right]\right\}+o(1) \qquad \text{as} \quad k\to\infty. \tag{6.2}$$

From (6.2) we get

$$\varliminf_{k\to\infty} S_k \geqslant \frac{1}{(q^2;q)_\infty}\left\{q^{1/4}\left|\sum_{j=0}^{2P}(-1)^j(2j+1)\,q^{j(j+1)}\right|\right.$$
$$\left.-\frac{q^{(2P+1)^2}}{1-q^{4P+2}}\left[4P+3-(4P+1)\,q^{4P+2}\right]\right\}.$$

Taking the limit as $P\to\infty$, we have

$$\varliminf_{k\to\infty} S_k \geqslant \frac{q^{1/4}}{(q^2;q)_\infty}\left|\sum_{j=0}^{\infty}(-1)^j(2j+1)\,q^{j(j+1)}\right|.$$

Now it is necessary to prove that

$$\sum_{j=0}^{\infty}(-1)^j(2j+1)\,q^{j(j+1)}\neq 0 \qquad \text{for} \quad 0<q\leqslant\beta_0.$$

Write $A_j(q)=(2j+1)\,q^{j(j+1)}$. Then $A_{j+1}(q)<A_j(q)$ for $j=0,1,\ldots$ if and only if

$$q^{2(j+1)}(2j+3)-2j-1<0, \qquad j=0,1,\ldots. \tag{6.3}$$

Inequality (6.3) holds for $0<q<\beta_0$ if it holds for $q=\beta_0$. It is easy to check that (6.3) holds if $q=\beta_0$ for $j=1,2,\ldots$. Thus if we write

$$\sum_{j=0}^{\infty}(-1)^j(2j+1)\,q^{j(j+1)}=1-3q^2+5q^6-7q^{12}+\sum_{j=4}^{\infty}(-1)^j A_j(q) \tag{6.4}$$

then the infinite series on the right side of (6.4) is positive for $0<q\leqslant\beta_0$ because $A_{j+1}(q)<A_j(q)$. The polynomial $1-3q^2+5q^6-7q^{12}$ is positive for $0<q\leqslant\beta_0$. We can now conclude that $\varliminf_{k\to\infty} S_k>0$. (Actually we have $\varliminf_{k\to\infty} S_k>\sum_{j=0}^{5} A_j(q)>0$.) ∎

Remark. The condition $0<q\leqslant\beta_0$ in Theorem 6.1 is not necessary. For example, if $P=10$ in (6.2) MATHEMATICA shows that the right side of (6.2) is positive for $0<q<.9$. Throughout this paper, however, we have tried to restrict ourselves, as much as possible, to arguments that avoid computer assisted proofs. It is certainly the case that Theorem 6.1 remains true for $0<q<1$ but we have no analytic proof.

Theorem 6.1 provides information about only one of the factors in μ_k, the factor that remains is $C_q(q^{1/2}\omega_k)$. It is not necessary to estimate this factor because the following identities allow us to eliminate it. The proofs are done by induction using the difference identities (2.5) and (2.6). Because the proofs are lengthy, we omit them.

THEOREM 6.2. *Define $P_n(z)$ and $Q_n(z)$ by*

$$P_n(z)=\sum_{j=0}^{n}\frac{(-1)^j\, q^{j(j+1)}(q^{1+n-j};\, q)_{2j+1}\, z^j}{(q;\, q)_{2j+1}}$$

$$Q_n(z)=\sum_{j=0}^{n}\frac{(-1)^j\, q^{j(j-1/2)}(q^{1+n-j};\, q)_{2j}\, z^j}{(q;\, q)_{2j}}.$$

Then, for $n=0, 1, 2, \ldots$

$$S_q(q^{n+1}\omega_k)=S_q(q\omega_k)\, P_n(\omega_k^2) \tag{6.5}$$

$$C_q(q^{n+1/2}\omega_k)=C_q(q^{1/2}\omega_k)\, Q_n(\omega_k^2). \tag{6.6}$$

The identities (6.5) and (6.6) can be used to eliminate the factor $C_q(q^{1/2}w_k)$ that occurs in μ_k. In effect, the factor divides out with a similar factor in a_k and b_k. To see this, we have

$$\begin{aligned} a_k &= \frac{1}{\mu_k}\int_{-1}^{1} f(t)\, C_q(q^{1/2}w_k t)\, d_q t \\ &= \frac{1}{S'_q(w_k)\, C_q(q^{1/2}w_k)}\sum_{j=0}^{\infty}\left[f(q^j)+f(-q^j)\right] C_q(q^{j+1/2}w_k)\, q^j. \end{aligned}$$

Now use (6.6) in the last expression to get

$$a_k=\frac{1}{S'_q(w_k)}\sum_{j=0}^{\infty}\left[f(q^j)+f(-q^j)\right] Q_j(w_k^2)\, q^j. \tag{6.7}$$

Similarly, for b_k by using (6.5) and the identity $S_q(qw_k)=-w_kC_q(q^{1/2}w_k)$ we have

$$b_k=\frac{-w_k}{S'_q(w_k)}\sum_{j=0}^{\infty}\left[f(q^j)-f(-q^j)\right] P_j(w_k^2)\, q^j. \tag{6.8}$$

While (6.7) and (6.8) have the disadvantage of not being expressible as q-integrals, they do show that the factor $C_q(q^{1/2}w_k)$ in μ_k may be divided out.

7. COMPLETENESS

An orthogonal system $\{\phi_n(x)\}$ is said to be complete if

$$\int_a^b f(x)\,\varphi_n(x)\,dx = 0, \qquad n = 0, 1, \ldots$$

implies that $f(x)$ is the zero function. Completeness is a fundamental property because it ensures uniqueness of Fourier coefficients and it also ensures that if

$$f(x) \sim \sum_{h=0}^{\infty} a_n \varphi_n(x),$$

and the series converges uniformly, then

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x).$$

In this section we will prove completeness of the system $C_q(q^{1/2}\omega_k z)$, $S_q(q\omega_k z)$ by applying a theorem of Phragmen–Lindelöf type. This theorem uses the concept of the order of a holomorphic function in a sector. Define $M_f(r, \alpha, \beta) = \max_{\alpha \leqslant \theta \leqslant \beta} |f(re^{i\theta})|$. Then the order ρ of $f(z)$ in the sector $\alpha \leqslant \theta \leqslant \beta$ is

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln^+ \ln^+ M_f(r, \alpha, \beta)}{\ln r}.$$

When the sector is the complex plane and $f(z)$ is entire, then p is simply the order of the entire function. The theorem to be applied is

THEOREM 7.1 [6]. *Let the function $f(z)$ be holomorphic inside an angle of opening π/α and continuous on the boundary. Assume that on the sides of the angle $|f(z)| \leqslant M$ and that the order ρ of the function is less than α. Then $|f(z)| \leqslant M$ throughout the angle.*

LEMMA 7.1. *Let $g(w)$ be bounded for $w = \pm q^k$, $k = 0, 1, \ldots$. Let $f(z) = C_q(q^{1/2}z) + iS_q(qz)$ and define $I(w)$ by*

$$I(w) = \int_{-1}^{1} g(z)\, f(wz)\, d_q z. \tag{7.1}$$

Then $I(w)$ is entire and has order zero.

Proof. The series expansion for $I(w)$ is

$$I(w) = \sum_{k=0}^{\infty} [g(q^k)+g(-q^k)]\, C_q(q^{k+1/2}w)\, q^k(1-q$$

$$+i \sum_{k=0}^{\infty} [g(q^k)-g(-q^k)]\, S_q(q^{k+1}w)\, q^k(1-q).) \tag{7.2}$$

Given any disk $|w| < R$, the functions $C_q(w)$ and $S_q(w)$ are bounded in $|w| < R$, say that $|C_q(w)| < M(R)$, $|S_q(w)| < M(R)$. If $|g(\pm q^k)| \leqslant B$, $k = 0, 1, \ldots$ then for $|w| < R$, (7.2) gives

$$|I(w)| \leqslant 4BM(R)(1-q) \sum_{k=0}^{\infty} q^k.$$

Thus the series in (7.2) converges uniformly in any disk $|w| < R$. Consequently $I(w)$ is entire. Next observe that

$$\max_{|w|=r} |I(w)| \leqslant \{\max_{|w|=r} |C_q(q^{1/2}w)| + \max_{|w|=r} |S_q(qw)|\} \cdot 2B.$$

That is,

$$\max_{|w|=r} |I(w)| \leqslant 4B \max_{|w|=r} \{|C_q(q^{1/2}w)|, |S_q(qw)|\},$$

and since

$$\overline{\lim_{r\to\infty}} \frac{\ln \ln \max_{|w|=r} |C_q(q^{1/2}w)|}{\ln r} = \overline{\lim_{r\to\infty}} \frac{\ln \ln \max_{|w|=r} |S_q(qw)|}{\ln r} = 0,$$

we have that

$$\overline{\lim_{r\to\infty}} \frac{\ln \ln \max_{|w|=r} |I(w)|}{\ln r} = 0$$

and $I(w)$ has order 0.

LEMMA 7.2. *Define $h(w) = I(w)/S_q(w))$. If $I(w)$ vanishes at the roots of $S_q(w)$ then $h(w)$ is entire of order zero.*

Proof. That $h(w)$ is entire follows from Lemma 7.1 and the fact that $I(w)$ vanishes at the roots of $S_q(w)$. The statement about order follows from Hademard's factorization theorem for entire functions [6].

Hadamard's theorem states that an entire function $\phi(z)$ of order ρ can be written as

$$\phi(z) = z^m[\exp \psi(z)]\, E(z; \phi),$$

where m is the order of the zero at $z = 0$, $\psi(z)$ is a polynomial of degree less than or equal to ρ, and $E(z; \phi)$ is the canonical product of the roots of $\phi(z)$. Since both $I(w)$ and $S_q(w)$ have order zero and since the roots of $S_q(w)$ are also roots of $I(w)$ we have

$$\frac{I(w)}{Sq(w)} = z^{m-1} E^*(z),$$

where m is the order of the root of $I(w)$ at $z = 0$ and $E^*(z)$ is a canonical product of roots of $I(w)$ that are not roots of $S_q(w)$ (if any). Thus $I(w)/S_q(w)$ has order zero.

THEOREM 7.2. *Let $f(w_k z) = C_q(q^{1/2} w_k z) + iS_q(q w_k z)$ where the w_k, $w_0 = 0 < w_1 < w_2 < \cdots$ are the roots of $S_q(z)$. Suppose that*

$$\int_{-1}^{1} g(z)\, f(w_k z)\, d_q z = 0, \qquad k = 0, 1, \ldots,$$

where $g(z)$ is bounded on $z = \pm q^k$, $k = 0, 1, \ldots$. Then $g(\pm q^k) = 0$, $k = 0, 1, 2, \ldots$.

Proof. The function $f(z)$ is entire of order 0. Take $\alpha = 1$ in Theorem 7.1 and take the angle to be the right half plane. The sides of the angle are then the imaginary axis. Consider the function

$$h(w) = \frac{\int_{-1}^{1} g(z)\, f(wz)\, d_q z}{S_q(w)}. \tag{7.3}$$

$h(w)$ is entire, of order zero by Lemmas 7.1 and 7.2. By Theorem 7.1, if $|h(w)| < M$ for w on the imaginary axis, then $|h(w)| < M$ for Re $w > 0$.

We will prove that $h(w)$ is bounded on the imaginary axis. First write (2.6) as

$$C_q(z) = \frac{q^{1/2}[S_q(q^{-1/2} z) - S_q(q^{1/2} z)]}{z}. \tag{7.4}$$

Setting $w = iy$ in the integrand of (7.1) and using the identity (7.4) gives

$$\frac{f(iyz)}{S_q(iy)} = \frac{S_q(iyz) - (1 + yz)\, S_q(qiyz)}{iyz S_q(iy)}. \tag{7.5}$$

Now note that if a, t are real with $|a| \leqslant 1$ then

$$|S_q(iat)| \leqslant |a|\, |S_q(it)|. \tag{7.6}$$

The variable z that appears in the q-integral in (7.1) and in (7.4) takes on values $\pm q^k$, $0 < q < 1$, so that $|z| \leqslant 1$. Then using (7.6) in (7.5) gives

$$\left| \frac{f(iyz)}{S_q(iy)} \right| \leqslant \frac{(1 + |1 + yz|)}{|y|}. \tag{7.7}$$

Since $h(w)$ is analytic in $|w| \leqslant 1$, it must be bounded there, say $|h(w)| < M$. We can then assume that $y > 1$ in (7.7). Since $|z| \leqslant 1$, the right side of (7.7) is bounded for $|y| > 1$, since $(1 + |1 + yz|/|y|) \leqslant (2/|y|) + |z| \leqslant (2/|y|) + 1 < 3$.

Then $|h(iy)| < M$ if $|y| \leqslant 1$, while if $y > 1$ then

$$|h(iy)| \leqslant \int_{-1}^{1} |g(z)| \frac{|f(iyz)|}{|S_q(iy)|}\, d_q z < 6B,$$

where $B = \text{lub}\, |g(z)|$, $z = \pm q^k$, $k = 0, 1, \ldots$. $h(w)$ is thus bounded on the imaginary axis and, by Theorem 7.1, $h(w)$ is bounded for $\operatorname{Re} w \geqslant 0$. Replacing w by $-w$ gives the same result for $\operatorname{Re} w \leqslant 0$. Thus $h(w)$ is a bounded entire function, and by Liouville's Theorem $h(w)$ is constant.

Next we prove that if $h(w) \equiv c$, then $c = 0$. This follows immediately with the observation that if w is real then $h(iw)$ is real while $h(w)$ is complex.

We now have that

$$\int_{-1}^{1} g(z)\, f(wz)\, d_q z \equiv 0, \tag{7.8}$$

and we must prove that $g(q^j) = 0$. Expand $f(wz)$ in power series using the power series for $C_q(z)$ and $S_q(z)$ to get

$$\begin{aligned}
&\int_{-1}^{1} g(z)\, f(wz)\, d_q z \\
&\quad = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \frac{[g(q^k) + g(-q^k)](-1)^n q^{(2n+1)k} q^{n(n+2)} w^{2n}}{(q, q^2; q^2)_n} \\
&\qquad + \frac{i}{1-q} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{[g(q^k) - g(-q^k)](-1)^n q^{(2n+2)k} q^{n^2+3n+1} w^{2n+1}}{(q, q^3; q^2)_n}.
\end{aligned} \tag{7.9}$$

Since (7.7) vanishes identically in w the identity theorem for analytic functions gives

$$\sum_{k=0}^{\infty} [g(q^k) + g(-q^k)]\, q^{(2n+1)k} = 0, \quad n = 0, 1, 2, \ldots \tag{7.10}$$

and

$$\sum_{k=0}^{\infty} [g(q^k)-g(-q^k)]\, q^{(2n+2)k}=0, \quad n=0, 1, 2, \dots . \tag{7.11}$$

Now (7.5) and (2.6) give $g(q^k)=g(-q^k)=0$, $k=0, 1, 2, \dots$. This completes the proof.

8. EXAMPLES

In this final section we give two simple examples of q-linear Fourier series.

(a) $f(x)=x$. This function is odd so the series contains only q-sine terms. We have

$$S_q(x; f)=-\frac{2}{q}\sum_{k=1}^{\infty}\frac{S_q(q\omega_k x)}{\omega_k S_q'(\omega_k)}. \tag{8.1}$$

(b) $f(x)=x^2$. This function is even so the series contains only q-cosine terms. We have

$$S_q(x; f)=\frac{1}{1+q+q^2}+2q^{-3/2}(1-q^2)\sum_{k=1}^{\infty}\frac{C_q(q^{1/2}\omega_k x)}{\omega_k^2 S_q'(\omega_k)} \tag{8.2}$$

We will prove convergence of the series in (8.1). Set $z=q\omega_k x$ and $\omega_k=q^{-k+(1/4)+\epsilon_k}$ in $S_q(z)$ to get

$$S_q(q\omega_k x)=\frac{q\omega_k x}{1-q}\, q^{-(k-(3/2)-\epsilon_k)^2}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n-k+(3/2)+\epsilon_k)^2} x^{2n}}{(q^2, q^3; q^2)_n}. \tag{8.3}$$

Denote the infinite series in (8.3) by $T(x, k, q)$. Then

$$\begin{aligned}|T(x, k, q)| &\leqslant \frac{1}{(q^2, q^3; q^2)_\infty}\sum_{n=0}^{\infty} q^{(n-k+(3/2)+\epsilon_k)^2}\\ &< \frac{1}{(q^2, q^3; q^2)_\infty}\left\{\sum_{n=0}^{\infty} q^{(n+3/2)^2}+\sum_{n=1}^{\infty} q^{(n-3/2-\epsilon_k)^2}\right\}.\end{aligned}$$

Since $0 < \epsilon_k < 1$,

$$\sum_{n=1}^{\infty} q^{(n-(3/2)-\epsilon_k)^2} < \sum_{k=1}^{\infty} q^{n^2-5n+9/4}.$$

Thus we have, writing

$$\begin{aligned} A(q) &= \sum_{n=0}^{\infty} q^{(n+3/2)^2} + \sum_{n=1}^{\infty} q^{n^2-5n+9/4}, \\ |S_q(q\omega_k x)| &\leqslant \frac{q\omega_k}{1-q}\, q^{-(k-3/2-\epsilon_k)^2} A(q), \end{aligned} \tag{8.4}$$

$|x| \leqslant 1$, $0 < q \leqslant \beta_0$.

From Theorem 7.1 we have then

$$\frac{2}{q}\, \frac{|S_q(q\omega_k x)|}{\omega_k\, |S_q'(\omega_k)|} = O(q^{2k}) \tag{8.5}$$

for $|x| \leqslant 1$, $0 < q \leqslant \beta_0$.

Hence the series (8.1) converges uniformly for $|x| \leqslant 1$, $0 < q \leqslant \beta_0$ and the sum is analytic in the unit disk. By completeness, the infinite series converges to $f(x) = x$ where $x = \pm q^j$, $j = 0, 1, \ldots$. Then by the identity theorem for analytic functions the series converges to x for all finite x. Thus

$$x = -\frac{2}{q} \sum_{k=1}^{\infty} \frac{S_q(q\omega_k x)}{\omega_k S_q'(\omega_k)}, \qquad 0 < q \leqslant \beta_0. \tag{8.6}$$

Remark. Equation (8.6) certainly must hold for $0 < q < 1$. However we are limited here because of the condition $0 < q \leqslant \beta_0$ in Theorem 6.1. A similar argument applied to the series in (7.2) establishes

$$x^2 = \frac{1}{1+q+q^2} + 2q^{-3/2}(1-q^2) \sum_{k=1}^{\infty} \frac{C_q(q^{1/2}\omega_k x)}{\omega_k^2 S_q'(\omega_k)}.$$

The argument can be extended to arbitrary polynomials.

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